On a nonlocal multivalued problem in an Orlicz-Sobolev space via Krasnoselskii's genus

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Abstract

This paper is concerned with the multiplicity of nontrivial solutions in an Orlicz-Sobolev space for a nonlocal problem involving N-functions and theory of locally Lispchitz continuous functionals. More precisely, in this paper, we study a result of multiplicity to the following multivalued elliptic problem:

$$\begin{cases} -M\left(\int_{\Omega} \Phi(|\nabla u|)dx\right) div\big(\phi(|\nabla u|)\nabla u\big) - \phi(|u|)u \in \partial F(u) \text{ in } \Omega, \\ u \in W_0^1 L_\Phi(\Omega), \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, $N \geq 2$, M is continuous function, Φ is an N-function with $\Phi(t) = \int_0^{|t|} \phi(s)s \ ds$ and $\partial F(t)$ is a generalized gradient of $F(t)$. We use genus theory to obtain the main result.

1 Introduction

The purpose of this article is investigate the multiplicity of nontrivial solutions to the multivalued elliptic problem

$$(P) \quad \begin{cases} -M\left(\int_{\Omega} \Phi(|\nabla u|)dx\right) div\big(\phi(|\nabla u|)\nabla u\big) - \phi(|u|)u \in \partial F(u) \text{ in } \Omega, \\ u \in W_0^1 L_\Phi(\Omega), \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain with $N \geq 2$, $F(t) = \int_0^t f(s)ds$ and

$$\partial F(t) = \left\{ s \in \mathbb{R}; F^0(t;r) \geq sr, \ r \in \mathbb{R} \right\}.$$

Here $F^0(t;r)$ denotes the generalized directional derivative of $t \mapsto F(t)$ in direction of r, that is,

$$F^0(t;r) = \limsup_{h \to t, s \downarrow 0} \frac{F(h+sr) - F(h)}{s}.$$

We shall assume in this work that $f(t)$ is locally bounded in $\mathbb{R}$ and

$$\underline{f}(t) = \lim_{\epsilon \downarrow 0} \text{ess inf} \left\{ f(s); |s - t| < \epsilon \right\} \text{ and } \overline{f}(t) = \lim_{\epsilon \downarrow 0} \text{ess sup} \left\{ f(s); |s - t| < \epsilon \right\}.$$

It is well known that

$$\partial F(t) = [\underline{f}(t), \overline{f}(t)], \text{ (see [13])},$$

and that, if $f(t)$ is continuous then $\partial F(t) = \{f(t)\}$.

Problem (P) with $\phi(t) = 2$, that is,

$$(*) \qquad \begin{cases} -M\left(\int_{\Omega} |\nabla u|^2 \, dx\right) \Delta u - u \in \partial F(u) \text{ in } \Omega, \\ u \in H_0^1(\Omega) \end{cases}$$

is called nonlocal because of the presence of the term $M\left(\int_{\Omega} |\nabla u|^2 dx\right)$ which implies that the equation $(*)$ is no longer a pointwise identity.

The reader may consult [3], [2], [23] and the references therein, for more information on nonlocal problems.

On the other hand, in this study, the nonlinearity f can be discontinuous. There is by now an extensive literature on multivalued equations and we refer the reader to [4], [20], [6], [5], [11], and references therein. The interest in the study of nonlinear partial differential equations with discontinuous nonlinearities has increased because many free boundary problems arising in mathematical physics may be stated in this form.

Among these problems, we have the obstacle problem, the seepage surface problem, and the Elenbaas equation, see for example [13], [14] and [15].

For enunciate the main result, we need to give some hypotheses on the functions M, ϕ and f.

The hypotheses on the function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ of C^1 class are the following:

(ϕ_1) For all $t > 0$,
$$\phi(t) > 0 \quad \text{and} \quad (\phi(t)t)' > 0.$$

(ϕ_2) There exist $l, m \in (1, N)$, $l \leq m < l^* = \dfrac{lN}{N - l}$ such that

$$l \leq \frac{\phi(t)t^2}{\Phi(t)} \leq m,$$

for $t > 0$, where $\Phi(t) = \displaystyle\int_0^{|t|} \phi(s)sds$.

The hypothesis on the continuous function $M : \mathbb{R}^+ \to \mathbb{R}^+$ is the following:

(M_1) There exist $k_0, k_1, \alpha, q_0, q_1 > 0$ and $b : \mathbb{R} \to \mathbb{R}$ of C^1 class such that

$$k_0 t^\alpha \leq M(t) \leq k_1 t^\alpha,$$

$\alpha > \frac{q_1}{l}$, where

$$m < q_0 \leq \frac{b(t)t^2}{B(t)} \leq q_1 < l^*,$$

for all $t > 0$ with

$$(b(t)t)' > 0, \; t > 0$$

and

$$B(t) = \int_0^t b(s)sds.$$

The hypotheses on the function $f : \mathbb{R} \to \mathbb{R}$ are the following:

(f_1) For all $t \in \mathbb{R}$,

$$f(t) = -f(-t).$$

(f_2) There exist $b_0, b_1 > 0$ and $a_0 \geq 0$ such that

$$b_0 b(t)t \leq f(t) \leq b_1 b(t)t, \; |t| \geq a_0.$$

(f_3) There exists $a_0 \geq 0$ such that

$$f(t) = 0, \; |t| \leq a_0.$$

The main result of this paper is:

Theorem 1.1 *Assume that conditions (ϕ_1), (ϕ_2), (M_1), $(f_1) - (f_3)$ hold. Then for $a_0 > 0$ sufficiently small (or $a_0 = 0$), the problem (P) has infinitely many solutions.*

Below we show two graphs of functions that satisfy the hypotheses $(f_1)-(f_3)$. Note that the second graph corresponds to a function that has an enumerable number of points of discontinuity.
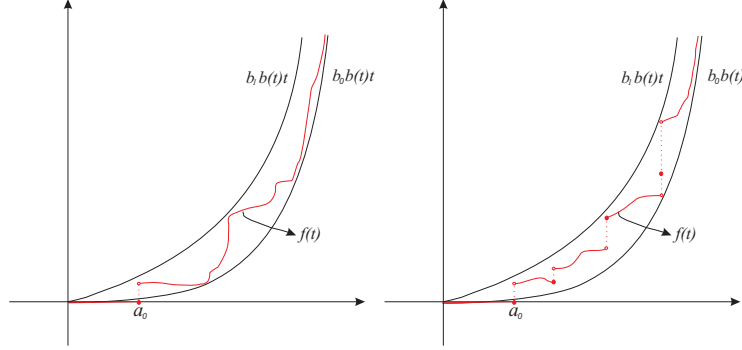
In the last twenty years the study on nonlocal problems of the type

$$(K) \qquad \begin{cases} -M\left(\displaystyle\int_{\Omega} |\nabla u|^2 \, dx\right)\Delta u = f(x, u) \text{ in } \Omega, \\ u \in H_0^1(\Omega) \end{cases}$$

grew exponentially. That was, probably, by the difficulties existing in this class of problems and that do not appear in the study of local problems, as well as due to their significance in applications. Without hope of being thorough, we mention some articles with multiplicity results and that are related with our main result. We will restrict our comments to the works that have emerged in the last four years

The problem (K) was studied in [23]. The version with p-Laplacian operator was studied in [19]. In both cases, the authors showed a multiplicity result using genus theory. In [27] the authors showed a multiplicity result for the problem (K) using the Fountain theorem and the Symmetric Mountain Pass theorem. In all these articles the nonlinearity is continuous. The case discontinuous was studied in [20]. With a nonlinearity of the Heaviside type the authors showed a existence of two solutions via Mountain Pass Theorem and Ekeland's Variational Principle.

In this work we extend the studies found in the papers above in the following sense:
a) We cannot use the classical Clark's Theorem for C^1 functional (see [21, Theorem 3.6]), because in our case, the energy functional is only locally Lipschitz continuous. Thus, in all section 5 we adapt for nondifferentiable functionals an argument found in [8].

b) Unlike [20], we show a result of multiplicity using genus theory considering a nonlinearity that can have a number enumerable of discontinuities.

c) Problem (P) possesses more complicated nonlinearities, for example:

(i) $\Phi(t) = t^{p_0} + t^{p_1}$, $1 < p_0 < p_1 < N$ and $p_1 \in (p_0, p_0^*)$.

(ii) $\Phi(t) = (1 + t^2)^\gamma - 1$, $\gamma \in (1, \frac{N}{N-2})$.

(iii) $\Phi(t) = t^p \log(1 + t)$ with $1 < p_0 < p < N - 1$, where $p_0 = \frac{-1 + \sqrt{1 + 4N}}{2}$.

(iv) $\Phi(t) = \int_0^t s^{1-\alpha} \left(\sinh^{-1} s\right)^\beta ds$, $0 \leq \alpha \leq 1$, $\beta > 0$.

d) We work with Orlicz-Sobolev spaces and some different estimates from those found in the papers above are necessary. For example, the Lemma 3.1 is a version for Orlicz-Sobolev spaces of a well-known result of Chang (see [13], [17] and [20, Lemma 3.3]). In the Lemma 5.2 one different estimate was necessary because of the presence of the nonlocal term.

The paper is organized as follows. In the next section we present a brief review on Orlicz-Sobolev spaces. In section 3 we recall some definitions and basic results on the critical point theory of locally Lipschitz continuous functionals. We also present variational tools which we will prove the main result of this paper. Furthermore, in this chapter, we prove the Lemma 3.1, which is a version for Orlicz-Sobolev spaces of a well-known result of Chang (see [13], [17] and [20, Lemma 3.3]). In Section 4 we present just some preliminary results involving genus theory that will be used in this work. In the Section 5 we prove Theorem 1.1.

2   A brief review on Orlicz-Sobolev spaces

Let ϕ be a real-valued function defined $[0, \infty)$ and having the following properties:

a) $\phi(0) = 0$, $\phi(t) > 0$ if $t > 0$ and $\lim_{t \to \infty} \phi(t) = \infty$.

b) ϕ is nondecreasing, that is, $s > t$ implies $\phi(s) \geq \phi(t)$.

c) ϕ is right continuous, that is, $\lim_{s \to t^+} \phi(s) = \phi(t)$.

Then, the real-valued function Φ defined on $\mathbb{R}$ by

$$\Phi(t) = \int_0^{|t|} \phi(s) \, ds$$

is called an N-function. For an N-function Φ and an open set $\Omega \subseteq \mathbb{R}^N$, the Orlicz space $L_\Phi(\Omega)$ is defined (see [1]). When Φ satisfies Δ_2-condition, that is, when there are $t_0 \geq 0$ and $K > 0$ such that $\Phi(2t) \leq K\Phi(t)$, for all $t \geq t_0$, the space $L_\Phi(\Omega)$ is the vectorial space of the measurable functions $u : \Omega \to \mathbb{R}$ such that

$$\int_\Omega \Phi(|u|) \, dx < \infty.$$

The space $L_\Phi(\Omega)$ endowed with Luxemburg norm, that is, the norm given by

$$|u|_\Phi = \inf\left\{\lambda > 0 : \int_\Omega \Phi\left(\frac{|u|}{\lambda}\right) dx \leq 1\right\},$$

is a Banach space. The complement function of Φ, denoted by $\widetilde{\Phi}$, is given by the Legendre transformation, that is

$$\widetilde{\Phi}(s) = \max_{t \geq 0}\{st - \Phi(t)\} \ \text{ for } \ s \geq 0.$$

These Φ and $\widetilde{\Phi}$ are complementary each other. Involving the functions Φ and $\widetilde{\Phi}$, we have the Young's inequality given by

$$st \leq \Phi(t) + \widetilde{\Phi}(s).$$

Using the above inequality, it is possible to prove the following Hölder type inequality

$$\left| \int_\Omega uv \ dx \right| \leq 2|u|_\Phi |v|_{\widetilde{\Phi}} \ \forall \ u \in L_\Phi(\Omega) \ \text{ and } \ v \in L_{\widetilde{\Phi}}(\Omega).$$

Hereafter, we denote by $W_0^1 L_\Phi(\Omega)$ the Orlicz-Sobolev space obtained by the completion of $C_0^\infty(\Omega)$ with norm

$$\|u\|_\Phi = |u|_\Phi + |\nabla u|_\Phi.$$

When Ω is bounded, there is $c > 0$ such that

$$|u|_\Phi \leq c|\nabla u|_\Phi.$$

In this case, we can consider

$$\|u\|_\Phi = |\nabla u|_\Phi.$$

Another important function related to function Φ, is the Sobolev conjugate function Φ_* of Φ defined by

$$\Phi_*^{-1}(t) = \int_0^t \frac{\Phi^{-1}(s)}{s^{(N+1)/N}} ds, \ t > 0.$$

The function Φ_* is very important because it is related to some embedding involving $W_0^1 L_\Phi(\Omega)$.

We say that Ψ increases essentially more slowly than Φ_* near infinity when

$$\lim_{t \to \infty} \frac{\Psi(kt)}{\Phi_*(t)} = 0, \ \text{ for all } k > 0.$$

Let Ω be a smooth bounded domain of $\mathbb{R}^N$. If Ψ is any N-function increasing essentially more slowly than Φ_* near infinity, then the imbedding $W_0^1 L_\Phi(\Omega) \hookrightarrow L_\Psi(\Omega)$ exists and is compact (see [1]).

The hypotheses $(\phi_1) - (\phi_2)$ implies that Φ, $\widetilde{\Phi}$, Φ_* and $\widetilde{\Phi}_*$ satisfy Δ_2-condition. This condition allows us conclude that:

1) $u_n \to 0$ in $L_\Phi(\Omega)$ if, and only if, $\int_\Omega \Phi(u_n) \ dx \to 0$.

2) $L_\Phi(\Omega)$ is separable and $\overline{C_0^\infty(\Omega)}^{|.|_\Phi} = L_\Phi(\Omega)$.
3) $L_\Phi(\Omega)$ is reflexive and its dual is $L_{\widetilde{\Phi}}(\Omega)$(see [1]).

Under assumptions $(\phi_1) - (\phi_2)$, some elementary inequalities listed in the following lemmas are valid. For the proofs, see [24].

Lemma 2.1 Let $\xi_0(t) = \min\{t^l, t^m\}$, $\xi_1(t) = \max\{t^l, t^m\}$, $\xi_2(t) = \min\{t^{l^*}, t^{m^*}\}$, $\xi_3(t) = \max\{t^{l^*}, t^{m^*}\}$, $t \geq 0$. Then

$$\xi_0(\|u\|_\Phi) \leq \int_\Omega \Phi(|\nabla u|) \ dx \ \leq \xi_1(\|u\|_\Phi),$$

$$\xi_2(|u|_{\Phi_*}) \leq \int_\Omega \Phi_*(|u|) \ dx \ \leq \xi_3(|u|_{\Phi_*})$$

and

$$\Phi_*(t) \geq \Phi_*(1)\xi_2(t).$$

Lemma 2.2 Let $\eta_0(t) = \min\{t^{q_0}, t^{q_1}\}, \eta_1(t) = \max\{t^{q_0}, t^{q_1}\}, t \geq 0$. Then

$$\eta_0(|u|_B) \leq \int_\Omega B(|u|)dx \leq \eta_1(|u|_B)$$

and

$$B(1)\eta_0(t) \leq B(t) \leq B(1)\eta_1(t), \ \ t \in \mathbb{R}.$$

Lemma 2.3 $\widetilde{\Phi}(\frac{\Phi(s)}{s}) \leq \Phi(s), \ s > 0.$

The next result is a version of Brezis-Lieb's Lemma [10] for Orlicz-Sobolev spaces and the proof can be found in [26].

Lemma 2.4 Let $\Omega \subset \mathbb{R}^N$ open set and $\Phi : \mathbb{R} \to [0, \infty)$ an N-function satisfies Δ_2-condition. If the complementary function $\widetilde{\Phi}$ satisfies Δ_2-condition, (f_n) is bounded in $L_\Phi(\Omega)$, such that

$$f_n(x) \to f(x) \ a.s \ x \in \Omega,$$

then

$$f_n \rightharpoonup f \ in \ L_\Phi(\Omega).$$

Corollary 2.1 The imbedding $W_0^1 L_\Phi(\Omega) \hookrightarrow L_B(\Omega)$ exists and is compact.

Proof: It is sufficiently to show that B increasing essentially more slowly than Φ_* near infinity. Indeed,

$$\frac{B(kt)}{\Phi_*(t)} \leq \frac{B(1)B(kt)}{\xi_2(t)} = B(1)k^{q_1}t^{q_1-l^*}, \ k > 0.$$

Since $q_1 < l^*$, we get

$$\lim_{t \to +\infty} \frac{B(kt)}{\Phi_*(t)} = 0.$$

■

3 Technical results on locally Lipschitz functional and variational framework

In this section, for the reader's convenience, we recall some definitions and basic results on the critical point theory of locally Lipschitz continuous functionals as developed by Chang [13], Clarke [17, 18] and Grossinho & Tersian [25].

Let X be a real Banach space. A functional $J : X \to \mathbb{R}$ is locally Lipschitz continuous, $J \in Lip_{loc}(X, \mathbb{R})$ for short, if given $u \in X$ there is an open neighborhood $V := V_u \subset X$ and some constant $K = K_V > 0$ such that

$$| J(v_2) - J(v_1) | \le K \parallel v_2 - v_1 \parallel, \ v_i \in V, \ i = 1, 2.$$

The directional derivative of J at u in the direction of $v \in X$ is defined by

$$J^0(u; v) = \limsup_{h \to 0, \ \sigma \downarrow 0} \frac{J(u + h + \sigma v) - I(u + h)}{\sigma}.$$

The generalized gradient of J at u is the set

$$\partial J(u) = \big\{ \mu \in X^*; \langle \mu, v \rangle \le J^0(u; v), \ v \in X \big\}.$$

Since $J^0(u; 0) = 0$, $\partial J(u)$ is the subdifferential of $J^0(u; 0)$. Moreover, $J^0(u; v)$ is the support function of $\partial J(u)$ because

$$J^0(u; v) = \max\{ \langle \xi, v \rangle; \xi \in \partial J(u) \}.$$

The generalized gradient $\partial J(u) \subset X^*$ is convex, non-empty and weak*-compact, and

$$m^J(u) = \min \big\{ \parallel \mu \parallel_{X^*}; \mu \in \partial J(u) \big\}.$$

Moreover,

$$\partial J(u) = \big\{ J'(u) \big\}, \text{if } J \in C^1(X, \mathbb{R}).$$

A critical point of J is an element $u_0 \in X$ such that $0 \in \partial J(u_0)$ and a critical value of J is a real number c such that $J(u_0) = c$ for some critical point $u_0 \in X$.

About variational framework, we say that $u \in W_0^1 L_\Phi(\Omega)$ is a weak solution of the problem (P) if it verifies

$$M \left(\int_\Omega \Phi(| \nabla u |) \ dx \right) \int_\Omega \phi(| \nabla u |) \nabla u \nabla v \ dx - \int_\Omega \phi(u) uv dx - \int_\Omega \rho v \ dx = 0,$$

for all $v \in W_0^1 L_\Phi(\Omega)$ and for some $\rho \in L_{\widetilde{B}}(\Omega)$ with

$$\underline{f}(u(x)) \le \rho(x) \le \overline{f}(u(x)) \ \text{ a.e in } \ \Omega,$$

and moreover the set $\{x \in \Omega; | u | \ge a_0\}$ has positive measure. Thus, weak solutions of (P) are critical points of the functional

$$J(u) = \widehat{M}\left(\int_\Omega \Phi(|\nabla u|)dx\right) - \int_\Omega \Phi(u)dx - \int_\Omega F(u)\ dx,$$

where $\widehat{M}(t) = \int_0^t M(s)ds$. In order to use variational methods, we first derive some results related to the Palais-Smale compactness condition for the problem (P).

We say that a sequence $(u_n) \subset W_0^1 L_\Phi(\Omega)$ is a Palais-Smale sequence for the locally lipschitz functional J associated of problem (P) if

$$J(u_n) \to c \text{ and } m^J(u_n) \to 0 \text{ in } (W_0^1 L_\Phi(\Omega))^*, \tag{3.1}$$

where

$$c = \inf_{\eta \in \Gamma} \max_{t \in [0,1]} J(\eta(t)) > 0$$

and

$$\Gamma := \{\eta \in C([0,1], X) : \eta(0) = 0, \ I(\eta(1)) < 0\}.$$

If (3.1) implies the existence of a subsequence $(u_{n_j}) \subset (u_n)$ which converges in $W_0^1 L_\Phi(\Omega)$, we say that these one functionals satisfies the nonsmooth $(PS)_c$ condition.

Note that $J \in Lip_{loc}(W_0^1 L_\Phi(\Omega), \mathbb{R})$ and from convex analysis theory, for all $w \in \partial J(u)$,

$$\langle w, v \rangle = M\left(\int_\Omega \Phi(|\nabla u|)\ dx\right)\int_\Omega \phi(|\nabla u|)\nabla u \nabla v\ dx - \int_\Omega \phi(u)uvdx - \langle \rho, v \rangle,$$

for some $\rho \in \partial\Psi(u)$, where $\Psi(u) = \int_\Omega F(u)dx$. We have $\Psi \in Lip_{loc}(L_B(\Omega), \mathbb{R})$, $\partial\Psi(u) \in L_{\widetilde{B}}(\Omega)$.

The next result is a version for Orlicz-Sobolev spaces of a well-known result of Chang (see [13], [17] and [20, Lemma 3.3]).

Lemma 3.1 *Suppose that M_1, (f_2) and (f_3) hold. For each $u \in L_B(\Omega)$, if $\rho \in \partial\Psi(u)$, then*

$$\underline{f}(u(x)) \leq \rho(x) \leq \overline{f}(u(x)) \text{ a.e } x \in \Omega,$$

and if $a_0 > 0$

$$\rho(x) = 0 \text{ a.e } x \in \{x \in \Omega; |\ u(x)\ |< a_0\}.$$

Proof: Considering $u, v \in L_B(\Omega)$, from definition

$$
\begin{aligned}
\Psi^0(u; v) &= \limsup_{h \to 0, t \to 0^+} \frac{\Psi(u + h + tv) - \Psi(u + h)}{t} \\
&= \limsup_{h \to 0, t \to 0^+} \frac{1}{t}\int_\Omega (F(u + h + tv) - F(u + h))\ dx.
\end{aligned}
$$

We set $(h_n) \subset L_B(\Omega)$ and $(t_n) \subset \mathbb{R}_+$ such that $h_n \to 0$ in $L_B(\Omega)$ and $t_n \to 0^+$. Thus,

$$\Psi^0(u; v) = \limsup_{n \to +\infty} \int_\Omega \frac{F(u + h_n + t_n v) - F(u + h_n)}{t_n} dx. \qquad (3.2)$$

Note that from the Mean Value Theorem, (M_1), (f_2) and (f_3) that,

$$F_n(u, v) := \frac{F(u + h_n + t_n v) - F(u + h_n)}{t_n} \le cb(|\theta_n(x)|)|\theta_n(x)||v|,$$

where

$$\theta_n(x) \in \left[\min\{u + h_n + t_n v, u + h_n\}, \max\{u + h_n + t_n v, u + h_n\} \right], \quad x \in \Omega.$$

Using monotonicity of $b(t)t$ we get

$$|F_n(u, v)| \le cb(|u + h_n + t_n v|)|u + h_n + t_n v||v| + cb(|u + h_n|)|u + h_n||v|.$$

On the other hand, by lemma 2.3 we have

$$\widetilde{B}(b(|u+h_n+t_nv|)|u+h_n+t_nv|) \le CB(|u+h_n+t_nv|) \le C\big(B(u)+B(h_n)+\eta_1(t_n)B(v)\big)$$

and

$$\widetilde{B}(b(|u + h_n + t_n v|)|u + h_n + t_n v|) \to \widetilde{B}(b(|u|)|u|) \quad a.e \;\; \text{in} \;\; \Omega,$$

where $\widetilde{B}(b(|u + h_n + t_n v|)|u + h_n + t_n v| \le cB(u) \in L^1(\Omega)$.

By Lebesgue's Theorem we obtain

$$\int_\Omega \widetilde{B}(b(|u + h_n + t_n v|)|u + h_n + t_n v|)dx \to \int_\Omega \widetilde{B}(b(|u|)|u|)dx.$$

From 2.4 we conclude that

$$\int_\Omega \widetilde{B}\left(b(|u + h_n + t_n v|)|u + h_n + t_n v| - b(|u|)|u|\right) dx \to 0.$$

Moreover, with obvious changes, we can prove that

$$\int_\Omega \widetilde{B}\left(b(|u + h_n|)|u + h_n| - b(|u|)|u|\right) dx \to 0.$$

Thus, by Fatou's lemma that

$$\limsup \int_\Omega F_n(u, v) \; dx \le \int_\Omega \limsup F_n(u, v) \; dx. \qquad (3.3)$$

From (3.2) and (3.3) we get

$$\Psi^0(u, v) \le \int_\Omega F^0(u, v) \; dx = \int_\Omega \max\{\langle \xi, v \rangle; \xi \in \partial F(u)\}dx.$$

Consider $\widehat{\rho} \in \partial\Psi(u) \subset L_B(\Omega)^* \equiv L_{\widetilde{B}}(\Omega)$ with $u \in L_B(\Omega)$. Then, there is $\rho \in L_{\widetilde{B}}(\Omega)$ such that

$$\langle \widehat{\rho}, v \rangle = \int_\Omega \rho v \ dx, \quad v \in L_B(\Omega).$$

We claim that

$$\rho(x) \geq \underline{f}(u(x)) \quad a.e \ \text{in} \ \Omega.$$

Arguing, by contradiction, we suppose that there is $A \subset \Omega$ with $|A| > 0$ such that $\rho(x) < \underline{f}(u(x))$. Hence,

$$\int_A \rho(x) \ dx < \int_A \underline{f}(u(x)) \ dx. \tag{3.4}$$

Let $v = -\chi_A$ be a function in $L_B(\Omega)$, where χ_A is characteristic function of set A. Thus,

$$-\int_A \rho \ dx = \int_\Omega \rho v \ dx \leq \Psi^0(u, v) \leq \int_\Omega \underline{f}(u(x))v dx = -\int_A \underline{f}(u(x)) \ dx,$$

with is a contradiction with (3.4). Thus

$$\rho(x) \geq \underline{f}(u(x)) \quad a.e \ \text{in} \ \Omega.$$

The inequality

$$\rho(x) \leq \overline{f}(u(x)) \quad a.e \ \text{in} \ \Omega$$

follows the same argument. ∎

4 Results involving genus

We will start by considering some basic notions on the Krasnoselskii genus that we will use in the proof of our main results.

Let E be a real Banach space. Let us denote by $\mathfrak{A}$ the class of all closed subsets $A \subset E \setminus \{0\}$ that are symmetric with respect to the origin, that is, $u \in A$ implies $-u \in A$.

Definition 4.1 *Let $A \in \mathfrak{A}$. The Krasnoselskii genus $\gamma(A)$ of A is defined as being the least positive integer k such that there is an odd mapping $\phi \in C(A, \mathbb{R}^k)$ such that $\phi(x) \neq 0$ for all $x \in A$. If k does not exist we set $\gamma(A) = \infty$. Furthermore, by definition, $\gamma(\emptyset) = 0$.*

In the sequel we will establish only the properties of the genus that will be used through this work. More information on this subject may be found in the references by [7], [12], [21] and [28].

Proposition 4.1 *Let $E = \mathbb{R}^N$ and $\partial\Omega$ be the boundary of an open, symmetric and bounded subset $\Omega \subset \mathbb{R}^N$ with $0 \in \Omega$. Then $\gamma(\partial\Omega) = N$.*

Corollary 4.1 *$\gamma(\mathcal{S}^{N-1}) = N$ where $\mathcal{S}^{N-1}$ is a unit sphere of $\mathbb{R}^N$.*

Proposition 4.2 *If $K \in \mathfrak{A}$, $0 \notin K$ and $\gamma(K) \geq 2$, then K has infinitely many points.*

5 Proof of Theorem 1.1

The plan of the proof is to show that the set of critical points of the functional J is compact, symmetric, does not contain the zero and has genus more than 2. Thus, our main result is a consequence of Proposition 4.2.

In the proof of the Theorem 1.1 we shall need the followings technical results:

Lemma 5.1 *The functional J is coercive.*

Proof: Using (M_1) and (f_2) we get

$$
\begin{aligned}
J(u) &\geq k_0 \int_0^{\int_\Omega \Phi(|\nabla u|)\,dx} s^\alpha ds - \int_\Omega \Phi(u)dx - b_1 \int_\Omega B(u)\,dx \\
&\geq \frac{k_0}{\alpha+1}\left(\int_\Omega \Phi(|\nabla u|)dx\right)^{\alpha+1} - \int_\Omega \Phi(u)dx - b_1\int_\Omega B(u)\,dx.
\end{aligned}
$$

From Lemmas 2.1 and 2.2 we obtain

$$
J(u) \geq \frac{k_0}{\alpha+1}\xi_0(\|u\|_\Phi)^{\alpha+1} - \xi_1(|u|_\Phi) - b_1\eta_1(|u|_B).
$$

Using now Corollary 2.1 we get the continuous imbedding $W_0^1 L_\Phi(\Omega) \hookrightarrow L_B(\Omega), L_\Phi(\Omega)$ hold. Hence, there are positive constants C_1, C_2 and C_3 such that, for $|\nabla u|_\Phi \geq 1$, we have

$$
J(u) \geq C_1\|u\|_\Phi^{l(\alpha+1)} - C_2\|u\|_\Phi^m - C_3\|u\|_\Phi^{q_1}.
$$

Since $l(\alpha+1) > q_1 > m$, we conclude that J is coercive. ∎

Now we prove that J satisfies the nonsmooth $(PS)_c$ condition.

Lemma 5.2 *The functional J satisfies the nonsmooth $(PS)_c$ condition, for all $c \in \mathbb{R}$.*

Proof: Let (u_n) be a sequence in $W_0^1 L_\Phi(\Omega)$ such that

$$
J(u_n) \to c \quad \text{and} \quad m^J(u_n) \to 0.
$$

From now we consider $(w_n) \subset \partial J(u_n) \subset (W_0^1 L_\Phi(\Omega))^*$ such that

$$
m^J(u_n) = \|w_n\|_* = o_n(1)
$$

and

$$
\langle w_n, v\rangle = M\left(\int_\Omega \Phi(|\nabla u_n|)\,dx\right)\int_\Omega \phi(|\nabla u_n|)\nabla u_n \nabla v\,dx - \int_\Omega \phi(u)uvdx - \langle \rho_n, v\rangle,
$$

with $\rho_n \in \partial\Psi(u_n)$.

Note that from Lemma 3.1 we have

$$\underline{f}(u_n) \le \rho_n \le \overline{f}(u_n) \quad \text{a.e in} \ \ \Omega.$$

On the other hand, since J is coercive, we derive that (u_n) is bounded in $W_0^1 L_\Phi(\Omega)$. Thus, passing to a subsequence, if necessary, we have

$$u_n \rightharpoonup u \ \ \text{in} \ \ W_0^1 L_\Phi(\Omega),$$

$$\frac{\partial u_n}{\partial x_i} \rightharpoonup \frac{\partial u}{\partial x_i} \ \ \text{in} \ \ L_\Phi(\Omega),$$

$$u_n \to u \ \ \text{in} \ \ L_B(\Omega) \ \text{and} \ L_\Phi(\Omega),$$

and

$$\int_\Omega \Phi(|\nabla u_n|) \ dx \to t_0 \ge 0.$$

If $t_0 = 0$, then from Lemma 2.1 we obtain

$$|\nabla u_n|_\Phi \le \xi_0^{-1} \left(\int_\Omega \Phi(|\nabla u_n|) \ dx \right) \to 0$$

and the proof is finished.

If $t_0 > 0$, since M is a continuous function, we get

$$M \left(\int_\Omega \Phi(|\nabla u_n|) \ dx \right) \to M(t_0).$$

Thus, from (M_1) and for n sufficiently large,

$$M \left(\int_\Omega \Phi(|\nabla u_n|) \ dx \right) \ge k_0 t_0^\alpha > 0. \tag{5.1}$$

Now we proof that (ρ_n) is bounded in $L_{\widetilde{B}}(\Omega)$. Note that from (f_2), (f_3) and (M_1) that

$$\overline{f}(t) \le cb(t)t.$$

Since

$$\overline{f}(t) = -\underline{f}(-t)$$

we get from Lemmas 2.2 and 2.3 that

$$\int_\Omega \widetilde{B}(u_n) \ dx \quad \le \quad C \int_{[u_n \ge 0]} B(u_n) \ dx + \int_{[u_n < 0]} \widetilde{B}(\underline{f}(-u_n)) \ dx$$

$$\le \quad \int_\Omega \widetilde{B}(u_n) \ dx \le C\eta(|u_n|_B) \le \overline{C}(\|u_n\|_\Phi),$$

which implies that (ρ_n) is bounded in $L_{\widetilde{B}}(\Omega)$.

Then

$$\int_\Omega \rho_n(u_n - u) \ dx \to 0. \tag{5.2}$$

From definition of (u_n) we have

$$o_n(1) = \langle w_n, u_n - u \rangle \quad = \quad M\left(\int_\Omega \Phi(|\nabla u_n|)\ dx\right) \int_\Omega \phi(|\nabla u_n|)\nabla u_n \nabla(u_n - u)\ dx$$
$$- \int_\Omega \phi(u_n)u_n(u_n - u)dx - \int_\Omega \rho_n(u_n - u)\ dx.$$

Since $|u_n - u|_\Phi$ goes to 0 and $(\phi(u_n)u_n)$ is bounded in $L_{\widetilde{\Phi}}(\Omega)$, have that

$$\int_\Omega \phi(u_n)u_n(u_n - u)dx \to 0. \tag{5.3}$$

We get from (5.2) and (5.3) that

$$M\left(\int_\Omega \Phi(|\nabla u_n|)\ dx\right) \int_\Omega \phi(|\nabla u_n|)\nabla u_n \nabla(u_n - u)\ dx \to 0.$$

From (5.1) and the last convergence implies that

$$\int_\Omega \phi(|\nabla u_n|)\nabla u_n \nabla(u_n - u)\ dx \to 0.$$

Setting $\beta : \mathbb{R}^N \to \mathbb{R}^N$ by

$$\beta(x) = \phi(|\nabla x|)\nabla x,\ x \in \mathbb{R}^N,$$

the last limit imply that for some subsequence, still denoted by itself,

$$(\beta(\nabla u_n(x)) - \beta(\nabla u(x)))(\nabla u_n(x) - \nabla u(x)) \to 0 \text{ a.e in } \Omega.$$

Applying a result found in Dal Maso and Murat [22], it follows that

$$\nabla u_n(x) \to \nabla u(x) \text{ a.e in } \Omega.$$

Then

$$u_n \to u \text{ in } W_0^1 L_\Phi(\Omega).$$

∎

Let K_c be the set of critical points of J. More precisely

$$K_c = \{u \in W_0^1 L_\Phi(\Omega) : 0 \in \partial J(u) \text{ and } J(u) = c\}.$$

Since J is even, we have that K_c is symmetric. The next result is important in our arguments and allows we conclude that K_c is compact. The proof can be found in [13].

Lemma 5.3 *If J satisfies the nonsmooth $(PS)_c$ condition, then K_c is compact.*

To prove that K_c does not contain zero, we construct a special class of the levels c.

For each $k \in \mathbb{N}$, we define the set

$$\Gamma_k = \{C \subset W_0^1 L_\Phi(\Omega) : C \text{ is closed}, C = -C \text{ and } \gamma(C) \geq k\},$$

and the values

$$c_k = \inf_{C \in \Gamma_k} \sup_{u \in C} J(u).$$

Note that

$$-\infty \leq c_1 \leq c_2 \leq c_3 \leq \dots \leq c_k \leq \dots$$

and, once that J is coercive and continuous, J is bounded below and, hence, $c_1 > -\infty$. In this case, arguing as in [9, Proposition 3.1], we can prove that each c_k is a critical value for the functional J.

Lemma 5.4 *Given $k \in \mathbb{N}$, there exists $\epsilon = \epsilon(k) > 0$ such that*

$$\gamma(J^{-\epsilon}) \geq k,$$

where $J^{-\epsilon} = \{u \in W_0^1 L_\Phi(\Omega) : J(u) \leq -\epsilon\}$.

Proof: Fix $k \in \mathbb{N}$, let X_k be a k-dimensional subspace of $W_0^1 L_\Phi(\Omega)$. Thus, there exists $C_k > 0$ such that

$$-C_k |\nabla u|_\Phi \geq -|u|_\Phi,$$

for all $u \in X_k$.

We now use the inequality above, (M_1), (f_2), (f_3), Lemmas 2.1 and 2.2 to conclude that

$$J(u) \leq \frac{k_1}{\alpha + 1} \xi_1(\|u\|_\Phi)^{\alpha+1} - \xi_0(C_k \|u\|_\Phi).$$

For $\|u\|_\Phi \leq 1$ we get

$$J(u) \leq \|u\|_\Phi^m \left(\frac{k_1}{\alpha + 1} \|u\|_\Phi^{(\alpha+1)l-m} - C_k^m \right).$$

Considering $R > 0$ such that

$$R < \min \left\{ 1, \left(\frac{\alpha + 1}{k_1} C_k^m \right)^{\frac{1}{(\alpha+1)l-m}} \right\},$$

there exists $\epsilon = \epsilon(R) > 0$ such that

$$J(u) < -\epsilon < 0,$$

for all $u \in \mathcal{S}_R = \{u \in X_k; |\nabla u|_\Phi = R\}$. Since X_k and $\mathbb{R}^k$ are isomorphic and $\mathcal{S}_R$ and S^{k-1} are homeomorphic, we conclude from Corollary 4.1 that $\gamma(\mathcal{S}_R) = \gamma(S^{k-1}) = k$. Moreover, once that $\mathcal{S}_R \subset J^{-\epsilon}$ and $J^{-\epsilon}$ is symmetric and closed, we have

$$k = \gamma(\mathcal{S}_R) \leq \gamma(J^{-\epsilon}).$$

■

Lemma 5.5 *Given $k \in \mathbb{N}$, the number c_k is negative.*

Proof: From Lemma 5.4, for each $k \in \mathbb{N}$ there exists $\epsilon > 0$ such that $\gamma(J^{-\epsilon}) \geq k$. Moreover, $0 \notin J^{-\epsilon}$ and $J^{-\epsilon} \in \Gamma_k$. On the other hand

$$\sup_{u \in J^{-\epsilon}} J(u) \leq -\epsilon.$$

Hence,

$$-\infty < c_k = \inf_{C \in \Gamma_k} \sup_{u \in C} J(u) \leq \sup_{u \in J^{-\epsilon}} J(u) \leq -\epsilon < 0.$$

■

A direct consequence of the last Lemma is that $0 \notin K_{c_k}$. The next result is also important in our arguments and the proof can be found in [13].

Lemma 5.6 *Suppose that X is a reflexive Banach space and J is even and a locally Lipschitz function, satisfying the $(PS)_c$ condition. If U is any neighborhood of K_c, then for any $\epsilon_0 > 0$ there exist $\epsilon \in (0, \epsilon_0)$ and a odd homeomorphism $\eta : X \to X$ such that:*
a) $\eta(x) = x$ for $x \notin J^{c+\epsilon} \backslash J^{c-\epsilon}$
b) $\eta(J^{c+\epsilon} \backslash U) \subset J^{c-\epsilon}$
c) If $K_c = \emptyset$, then $\eta(J^{c+\epsilon}) \subset J^{c-\epsilon}$.

Lemma 5.7 *If $c_k = c_{k+1} = ... = c_{k+r}$ for some $r \in \mathbb{N}$, then*

$$\gamma(K_{c_k}) \geq r + 1.$$

Proof: Suppose, by contradiction, that $\gamma(K_{c_k}) \leq r$. Since K_{c_k} is compact and symmetric, there exists a closed and symmetric set U with $K_{c_k} \subset U$ such that $\gamma(U) = \gamma(K_{c_k}) \leq r$. Note that we can choose $U \subset J^0$ because $c_k < 0$. By the deformation lemma 5.6 we have an odd homeomorphism $\eta : W_0^1 L_\Phi(\Omega) \to W_0^1 L_\Phi(\Omega)$ such that $\eta(J^{c_k+\delta} - U) \subset J^{c_k-\delta}$ for some $\delta > 0$ with $0 < \delta < -c_k$. Thus, $J^{c_k+\delta} \subset J^0$ and by definition of $c_k = c_{k+r}$, there exists $A \in \Gamma_{k+r}$ such that $\sup_{u \in A} J(u) < c_k + \delta$, that is, $A \subset J^{c_k+\delta}$ and

$$\eta(A - U) \subset \eta(J^{c_k+\delta} - U) \subset J^{c_k-\delta}. \tag{5.4}$$

But $\gamma(\overline{A-U}) \geq \gamma(A) - \gamma(U) \geq k$ and $\gamma(\eta(\overline{A-U})) \geq \gamma(\overline{A-U}) \geq k$. Then $\eta(\overline{A-U}) \in \Gamma_k$ and this contradicts (5.4). Hence, this lemma is proved. ■

5.1 Proof of Theorem 1.1

If $-\infty < c_1 < c_2 < ... < c_k < ... < 0$ and since each c_k critical value of J, then we obtain infinitely many critical points of J and hence, the problem (P) has infinitely many solutions.

On the other hand, if there are two constants $c_k = c_{k+r}$, then $c_k = c_{k+1} = \ldots = c_{k+r}$ and from Lemma 5.7, we have

$$\gamma(K_{c_k}) \geq r + 1 \geq 2.$$

From Proposition 4.2, K_{c_k} has infinitely many points.

Let (u_k) critical points of J. Now we show that, for

$$a_0 < \xi_1^{-1}\left(\frac{k_0 l}{\Phi(1) \mid \Omega \mid m^2}\xi_0(C)^{\alpha+1}\right), \tag{5.5}$$

we have that

$$\left\{x \in \Omega : \mid u_k(x)\mid \geq a_0\right\}$$

has positive measure. Thus every critical points of J, are solutions of (P). Suppose, by contradiction, that this set has null measure. Thus

$$\begin{aligned}
0 &= M\left(\int_\Omega \Phi(\mid \nabla u_k \mid)dx\right)\int_\Omega \phi(\mid \nabla u_k \mid)\mid \nabla u_k \mid^2 dx - \int_\Omega \phi(\mid u_k \mid)\mid u_k \mid^2 dx \\
&\geq k_0 l\left(\int_\Omega \Phi(\mid \nabla u_k \mid)dx\right)^{\alpha+1} - m\int_\Omega \Phi(u_k)dx \\
&\geq k_0 l\xi_0(\parallel u_k \parallel_\Phi)^{(\alpha+1)} - m\Phi(a_0)\mid \Omega \mid,
\end{aligned}$$

where we conclude

$$k_0 l\xi_0(\parallel u_k \parallel_\Phi)^{(\alpha+1)} \leq m\xi_1(a_0)\mid \Omega \mid \Phi(1). \tag{5.6}$$

Since $c_k \leq -\epsilon < 0$, there exists $C > 0$ such that $\|u_k\| \geq C > 0$. Hence

$$a_0 \geq \xi_1^{-1}\left(\frac{k_0 l}{\Phi(1) \mid \Omega \mid m^2}\xi_0(C)^{\alpha+1}\right),$$

which contradicts (5.5). Then,

$$\left\{x \in \Omega : \mid u_k(x)\mid \geq a_0\right\}$$

has positive measure. ∎

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